Inexact Newton Dogleg Methods

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Nonlinear problem: $F(x) = 0, \quad F : \mathbb{R}^n \to \mathbb{R}^n.$

Start with classical . . .

Newton's Method:

Given an initial x.

Iterate:

Solve $F'(x)s = -F(x)$.

Update $x \leftarrow x + s$.

Globalizations.

Idea: Repeat as necessary ...

- *Test* a step for acceptable progress.
- If unacceptable, *modify* it and test again.

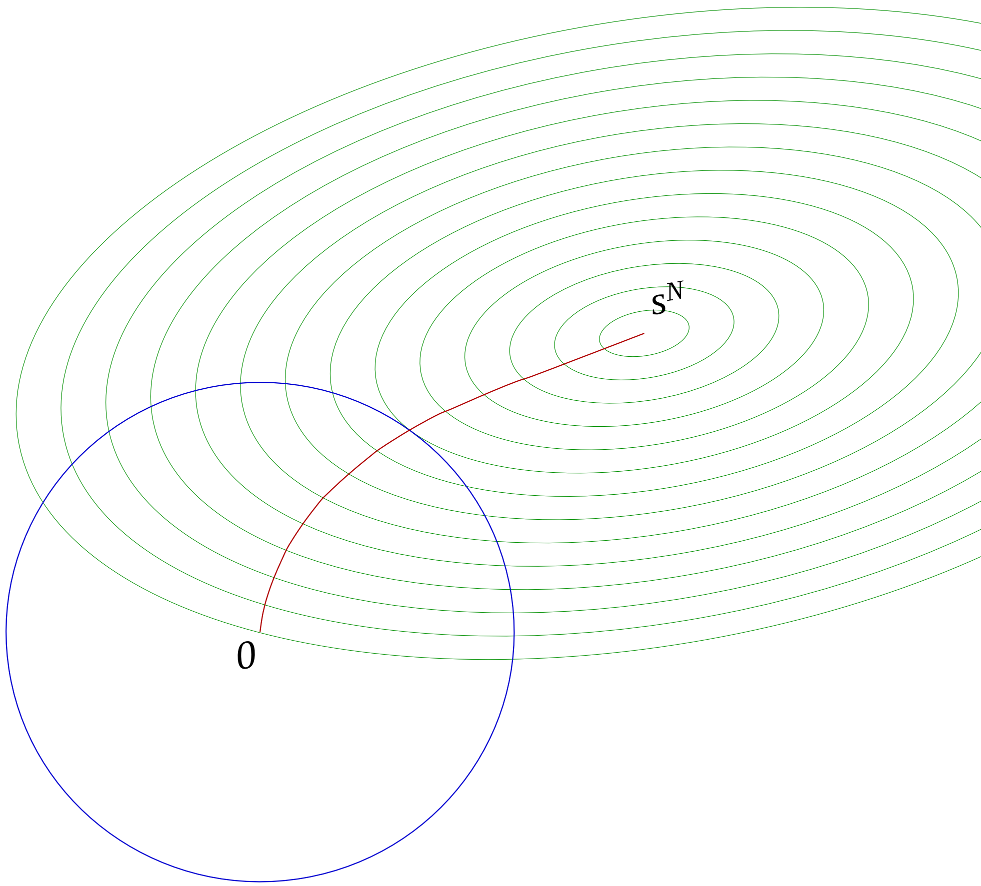
Major approaches:

- *Backtracking* (linesearch, damping).
- *Trust region*.

Trust region globalization.

- $s = \arg\min_{\|w\| \le \delta} \|F(u) + F'(u)\, w\|$.

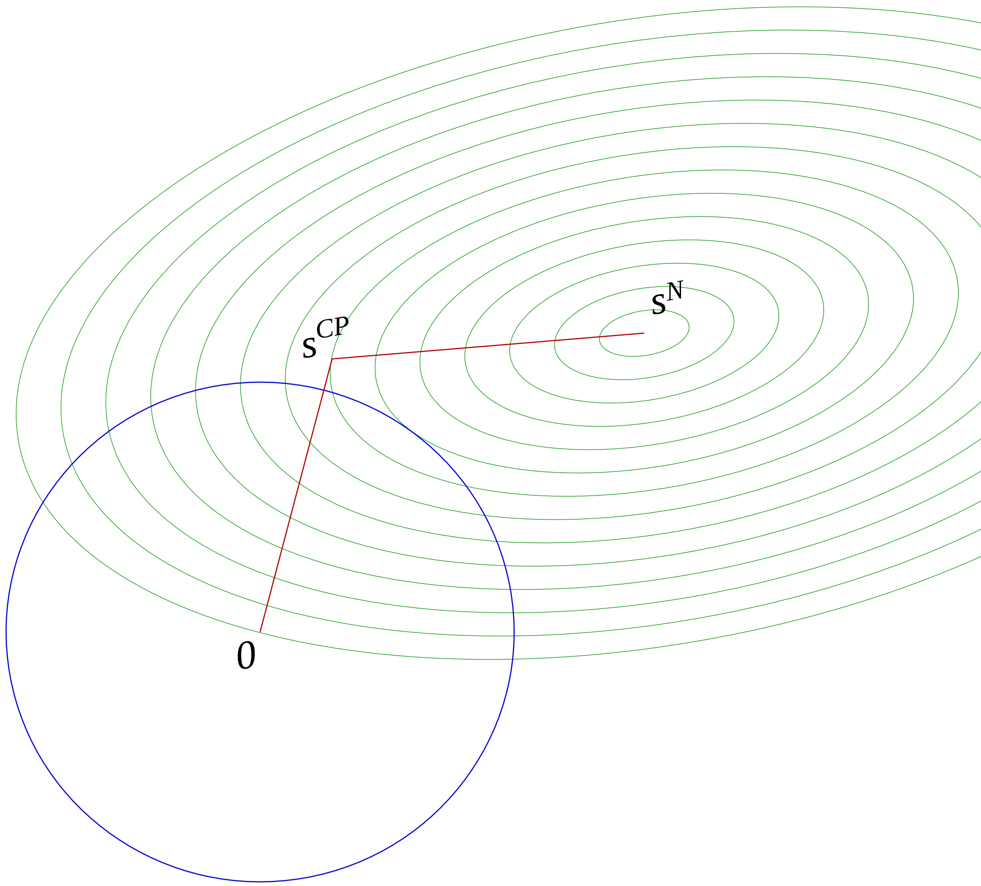
- Can't be computed exactly.



The dogleg step.

- Γ^{DL}: $0 \to s^{\mathrm{CP}} \to s^{N}$.

- $s = \underset{\|w\| \leq \delta, w \in \Gamma^{\mathrm{DL}}}{\arg\min} \|F(u) + F'(u)\, w\|$.



Work toward *inexact Newton* and *Newton–Krylov* adaptations.

Straightforward:

- $\|F(u) + F'(u)\, s^{IN}\| \le \eta \|F(u)\|$
- $\Gamma^{\rm DL}$: $0 \to s^{\rm CP} \to s^{IN}$.

Inexact Newton Dogleg Method:

Given $\eta_{\max} \in [0,1)$, $\delta_{\min} > 0$, $t \in (0,1)$, $0 < \theta_{\min} < \theta_{\max} < 1$,
and initial u and $\delta \geq \delta_{\min}$.

Iterate:

Choose $\eta \in [0, \eta_{\max}]$ and s^{IN} such that

$$\|F(u) + F'(u)\, s^{IN}\| \leq \eta \|F(u)\|.$$

Determine $s \in \Gamma^{\mathrm{DL}}$.

While $ared < t \cdot pred$ do:

Choose $\theta \in [\theta_{\min}, \theta_{\max}]$.

Update $\delta \longleftarrow \max\{\theta\delta, \delta_{\min}\}$.

Redetermine $s \in \Gamma^{\mathrm{DL}}$.

Update $u \leftarrow u + s$ and update δ.

- $ared \equiv \|F(u)\| - \|F(u+s)\|, \quad pred \equiv \|F(u)\| - \|F(u) + F'(u)\, s\|$.
- Choose θ, update δ a la Dennis–Schnabel.
- Determine $s \in \Gamma^{\mathrm{DL}}$ so that $\|s\| \geq \min\{\|s^{IN}\|, \delta_{\min}\}$.

Recall: u is a *stationary point* of $\|F\| \iff \|F(u)\| \leq \|F(u) + F'(u)\, s\| \ \forall s$.

Theorem: *Assume F is continuously differentiable. If u_* is a limit point of $\{u_k\}$, then u_* is a stationary point of $\|F\|$. If additionally $F'(u_*)$ is nonsingular, then $F(u_*) = 0$ and $u_k \to u_*$; furthermore, $s_k = s_k^{IN}$ is acceptable for all sufficiently large k.*

Proof: Since $\|F(u_k) + F'(u_k)\, s_k^{IN}\| \leq \eta_{\max}\|F(u_k)\|$ and $\|s_k\| \geq \min\{\|s_k^{IN}\|, \delta_{\min}\}$, one can show: If u_* is either a non-stationary point or such that $F'(u_*)$ is nonsingular, then there is an $\bar{\eta} < 1$ such that

$$\|F(u_k) + F'(u_k)\, s_k\| \leq \bar{\eta}\|F(u_k)\|$$

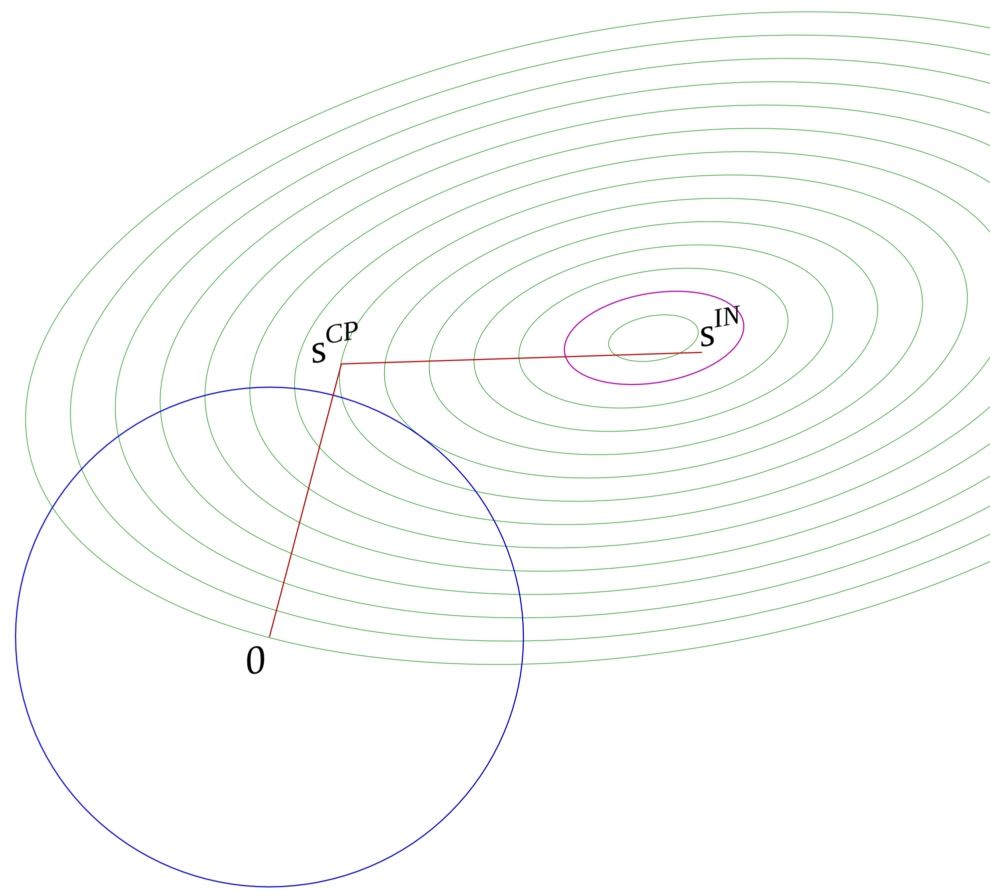
for u_k near u_*. The theorem follows from Eisenstat–W (1994), Cor. 3.6.

Possible **big** problem:

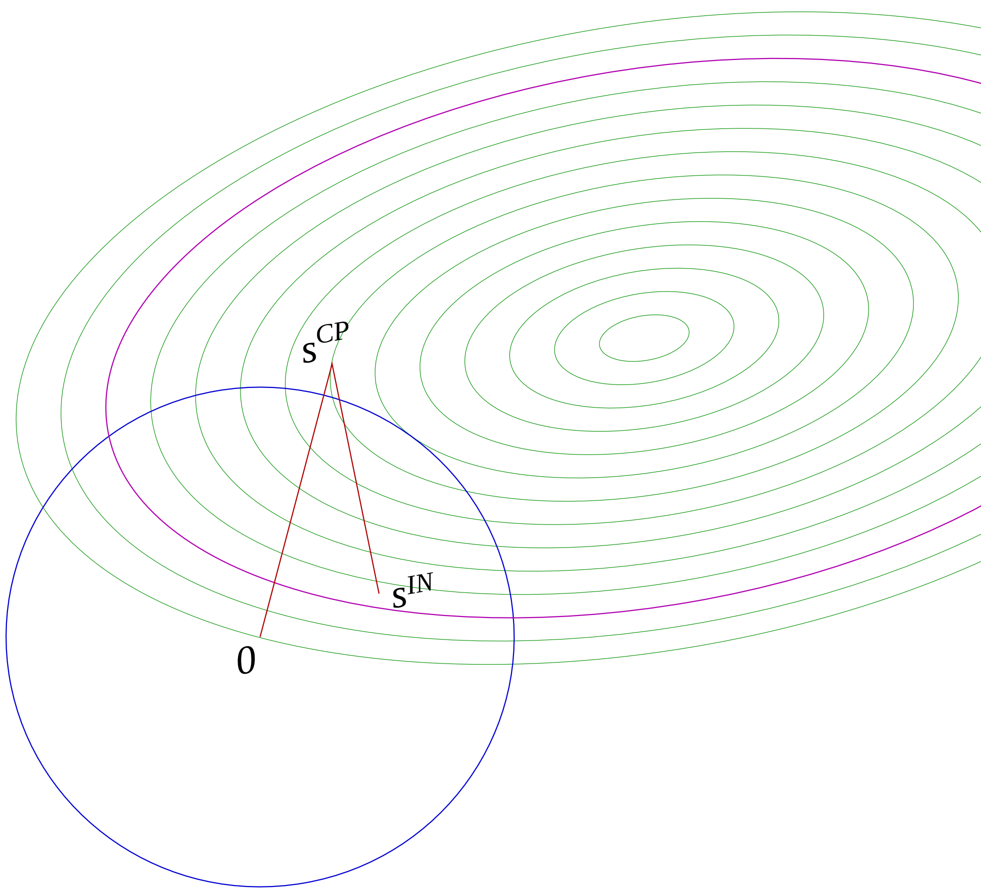
$$\boxed{\textit{Evaluating } s^{\mathrm{CP}} \textit{ requires } F'^T\textit{-products.}}$$

- ▷ Analytic evaluation may be expensive, infeasible.
- ▷ Finite-difference approximation won't work.
- ▷ Automatic differentiation?
- ▷ Brown–Saad (1990): dogleg-within-the-Krylov-subspace using (unrestarted) GMRES.
- ▷ Not a problem when $F' = F'^T$.

Minor consideration: For **any** $\eta \in [0, \eta_{\max})$, $\|F(u) + F'(u)\, s\|$ may not decrease monotonically along Γ^{DL}.



More serious consideration: Unless $\eta \in [0, \eta_{\max})$ is small (how small?), we may have $\langle s^{IN}, s^{\mathrm{CP}} \rangle < \|s^{\mathrm{CP}}\|^2$ or $\|s^{IN}\| < \|s^{\mathrm{CP}}\|$.



How to choose $s \in \Gamma^{\mathrm{DL}}$?

The Standard Strategy.

If $\|s^{IN}\| \le \delta$,

$$s = s^{IN}$$

Else if $\|s^{\mathrm{CP}}\| \ge \delta$,

$$s = (\delta / \|s^{\mathrm{CP}}\|) s^{\mathrm{CP}}$$

Else

$$s = (1-\gamma) s^{\mathrm{CP}} + \gamma s^{IN}$$

for $\gamma \in (0,1)$ such that $\|s\| = \delta$

- $s \in \Gamma^{\mathrm{DL}}$ is uniquely determined.
- s^{IN} is always computed; s^{CP} may not be.
- If η isn't small, we may have $s = s^{IN}$ when $s = \lambda s^{\mathrm{CP}}$ would be preferred.

An Alternative Strategy.

If $\|s^{\mathrm{CP}}\| \geq \delta$,

$$s = (\delta / \|s^{\mathrm{CP}}\|) s^{\mathrm{CP}}$$

Else if $\|F(u) + F'(u)\, s^{\mathrm{CP}}\| \leq \eta \|F(u)\|$,

$$s = s^{\mathrm{CP}}$$

Else if $\|s^{IN}\| \leq \delta$,

$$s = s^{IN}$$

Else

$$s = (1 - \gamma) s^{\mathrm{CP}} + \gamma s^{IN}$$

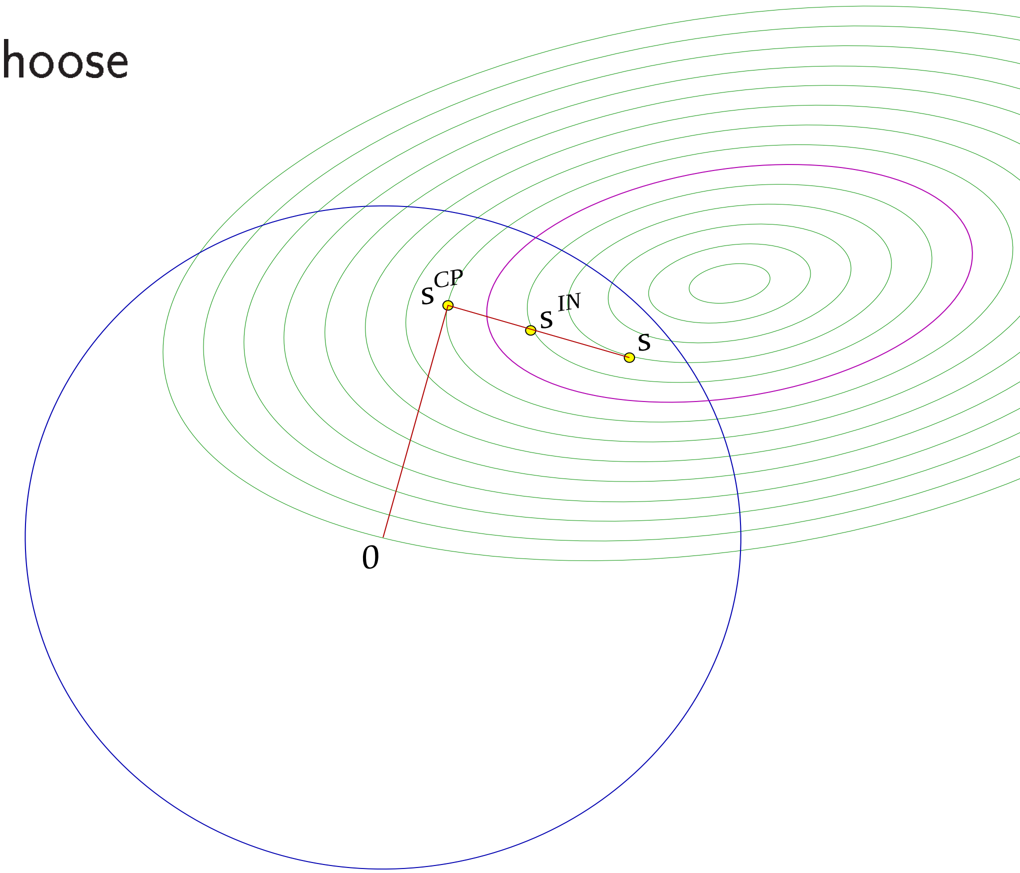
for $\gamma \in (0, 1)$ such that $\|s\| = \delta$

- $s \in \Gamma^{\mathrm{DL}}$ is uniquely determined.
- s^{CP} is always computed; s^{IN} may not be.
- s is appropriately biased toward s^{CP}.

Further refinements.

- If needed, s^{IN} can be computed as $s^{IN} = s^{\mathrm{CP}} + z$, where $\|r^{\mathrm{CP}} + F'(u)\,z\| \le \eta \|F(u)\|$ and $r^{\mathrm{CP}} \equiv F(u) + F'(u)\,s^{\mathrm{CP}}$.

- Having both s^{CP} and s^{IN}, we can choose $s = (1-\gamma)s^{\mathrm{CP}} + \gamma s^{IN}$ so that $\|s\| \le \delta$ and $\|F(u) + F'(u)\,s\|$ is minimal (easy).



Numerical experiments. *Extremely preliminary!!*

- IBM Linux cluster, 4 nodes (8 CPUs).
- MPSalsa + NOX.
- No row-sum scaling (yet).
- Alternative strategy computes $s^{IN} = s^{\mathrm{CP}} + z$, does not minimize $\|F(u) + F'(u)\, s\|$.

2D Thermal Convection Problem. Run times in seconds.

Ra	Backtracking (Quad.)	Dogleg Std.	Dogleg Alt.
10^3	57	56	56
10^4	111	94	93
10^5	146	147	98
10^6	409	1003	265
Geo. Means	139	167	108

Adaptive (Choice 1) Forcing Terms

Ra	Dogleg Std.	Dogleg Alt.
10^3	83	82
10^4	121	126
10^5	293	262
10^6	1266	1171
Geo. Means	247	237

Constant (10^{-4}) Forcing Terms

2D Backward Facing Step Problem. Run times in seconds.

Re	Dogleg Std.	Dogleg Alt.
100	20	23
200	48	36
300	163	30
400	210	35
500	F	63
600	F	137
700	F	F
750	F	F
800	F	F
Geo. Means*	95	28

Adaptive (Choice 1) Forcing Terms

$^{*}100 \leq Re \leq 400$

Re	Dogleg Std.	Dogleg Alt.
100	22	22
200	42	42
300	94	100
400	F	F
500	63	63
600	109	71
700	125	133
750	136	142
800	268	146
Geo. Means**	247	237

Constant (10^{-4}) Forcing Terms

$^{**}Re \neq 400$

Conclusions.

- **None yet!** Except . . .
- These dogleg methods can solve nontrivial problems.
- Methods, strategies, and refinements bear further study.